A fully relativistic twisted accretion disc around a Kerr black hole

V. Zhuravlev[1] P. Ivanov[2]

[1]Moscow State University

[2]Lebedev Physical Institute

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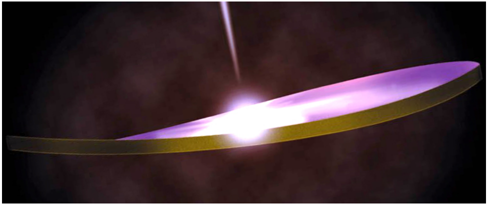
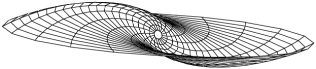


Overview of previous analytical studies



- Bardeen & Petterson (1975) — a first quantitative study, the disc is considered as a collection of "rigid" rings, the equations obtained are however quantitatively wrong, say, they do not conserve angular momentum.
- Petterson (1977,1978)
  Hatchett, Begelman & Sarazin (1981) — twisted coordinate systems were considered for Newtonian discs, the equations are improved but not quite...
- Papaloizou & Pringle (1983) — a first correct consideration. It was shown that one has to consider velocity/density perturbations in a self-consistent picture. It was stressed that these perturbations determine evolution/shapes of stationary configurations for nearly Keplerian, low viscosity discs.

Overview of previous analytical studies

- Papaloizou & Lin (1995) — bending waves, the case of low viscosity.
- Ivanov & Illarionov (1997) — stationary solution, low viscosity and post-Newtonian corrections, it was found that a low viscosity stationary disc exhibits radial oscillations instead of alignement with the equatorial plane in case of disk gas rotating in the same sense as the black hole.
- Demianski & Ivanov (1997) — a full system of equations with post-Newtonian corrections describing a low viscosity twisted disc.
- Lubow, Ogilvie & Pringle (2002) — time dependant evolution of such discs.

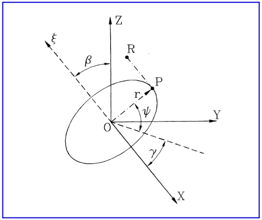
Overview of previous analytical studies

In order for the disc to have a stationary twisted shape the spherical symmetry of the problem should, in general, be broken and the disc plane at large distance from a centre of gravity should not coincide with a symmetry plane of the problem.

In the case of a black hole this is determined by its rotation, which leads to the presence of so-called gravitomagnetic force

$$F_{\xi} = \frac{4a\Omega M^2}{r^2}\beta \sin\psi,$$

which is balanced by hydrodynamical forces induced in the disc.



Here $|a| < 1$ is the black hole rotational parameter, $\Omega = \sqrt{M/r^3}$ is the Keplerian angular frequency of a free circular motion, it is assumed that $r \gg r_g$, and the natural system of units $c = G = 1$ is used hereafter.

Overview of previous analytical studies

The previous studies showed that a nearly Keplerian twisted disc has two different types of time evolution and stationary configurations for $\alpha > h/r$ and $\alpha < h/r$, where α is the well known Shakura-Sunyaev parameter.

h/r is assumed to be small, it is of the order of $10^{-2} - 10^{-3}$.

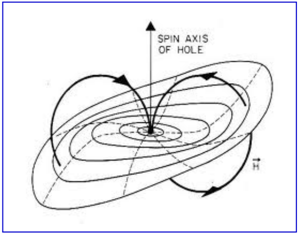
Overview of previous analytical studies

The case of $\alpha > h/r$

In this case a stationary twisted disc always aligns with the equatorial plane of a rotating black hole at radii smaller than or of the order of

$$R_1 \sim \alpha^{2/3} a^{2/3} (r/h)^{4/3} R_g,$$

where R_g is the gravitational radius, a is the rotational parameter of the black hole.



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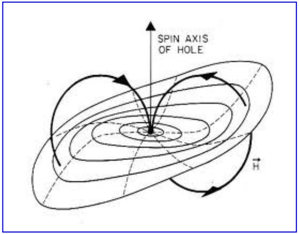


The time evolution has a character of diffusion, with characteristic time scale

$$t_1 = \alpha (r/h)^2 \Omega^{-1},$$

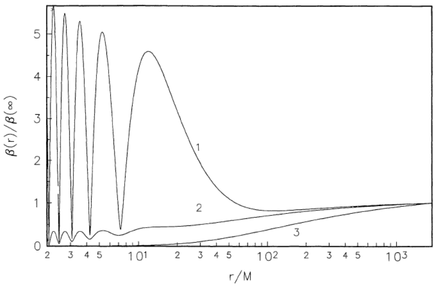
where the Keplerian angular frequency $\Omega = \sqrt{GM/r^3}$, M is the mass of the black hole.

Overview of previous analytical studies

The case of $\alpha < h/r$

When $a > 0$ a stationary twisted disc experiences radial oscillations of its inclination angle β (The angle between a unit vector normal to the disc's rings and the equatorial plane of the black hole). When $a < 0$ it aligns with the equatorial plane. The scale of oscillation/alignement

$$R_2 \sim a^{2/5}(r/h)^{4/5} R_g$$



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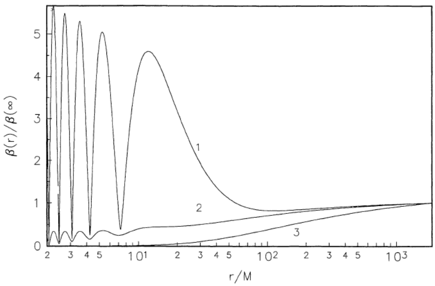


The time evolution is wave-like with a characteristic time scale of order of the "sound crossing" time scale

$$t_2 \sim (r/h)\Omega^{-1}$$

Metric

The Kerr metric of a slowly rotating black hole in the linear approximation in rotational parameter $a \ll 1$:

$$ds^2 = (1 - 2M/R)dt^2 - (1 - 2M/R)^{-1} dR^2 - R^2(d\theta^2 + sin^2\theta d\phi^2) + 4a\frac{M}{R} sin^2\theta \, d\phi \, dt$$

Let us change to the "isotropic" radial coordinate:

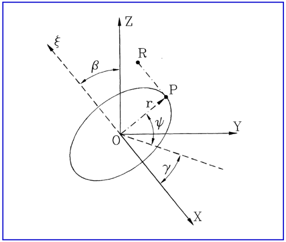
$$R = R_I \left(1 + \frac{M}{2R_I}\right)^2$$

and get a metric with the spacial line element proportional to the Cartesian spatial line element:

$$ds^2 = K_1^2 dt^2 + 2ar^2 K_1 K_3 d\phi dt - K_2^2(dr^2 + dz^2 + r^2 d\phi^2)$$

Twisted coordinates

$$\begin{pmatrix} \tau \\ r\cos\psi \\ r\sin\psi \\ \xi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma & 0 \\ 0 & -\sin\gamma & \cos\gamma & \beta \\ 0 & \beta\sin\gamma & -\beta\cos\gamma & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$



1. $|\beta(\tau, r)| \ll 1$, $\gamma(\tau, r)$ are functions to be determined
2. $\Psi_1 = \beta\cos\gamma$, $\Psi_2 = \beta\sin\gamma \quad \Longrightarrow \quad \mathbf{W} = \beta e^{i\gamma}$
3. change to $\varphi = \psi + \gamma$

An orthonormal tetrad

$$\mathbf{e}_\tau = \frac{1}{K_1}\left(\partial_\tau \right)$$

$$\mathbf{e}_r = \frac{1}{K_2}\left(\partial_r \right)$$

$$\mathbf{e}_\varphi = \frac{1}{K_2}\left(-ar\frac{K_3}{K_1} \partial_\tau + \frac{1}{r}\partial_\varphi \right)$$

$$\mathbf{e}_\xi = \frac{1}{K_2}\left(\partial_\xi\right)$$

An orthonormal tetrad

$$\mathbf{e}_\tau = \frac{1}{K_1}\left(\partial_\tau + \xi U \partial_r + \frac{\xi}{r}\partial_\varphi U \partial_\varphi - rU\partial_\xi\right)$$

$$\mathbf{e}_r = \frac{1}{K_2}\left(-a\xi\frac{K_3}{K_1}\partial_\varphi Z \partial_\tau + (1+\xi W)\partial_r + \frac{\xi}{r}\partial_\varphi W \partial_\varphi - rW\partial_\xi\right)$$

$$\mathbf{e}_\varphi = \frac{1}{K_2}\left(-ar\frac{K_3}{K_1}(1-\frac{\xi}{r}Z)\partial_\tau - a\xi\frac{K_3}{K_1} rU\partial_r + \left(1 - ar\xi\frac{K_3}{K_1}\partial_\varphi U\right)\frac{1}{r}\partial_\varphi + ar\frac{K_3}{K_1} rU\partial_\xi\right)$$

$$\mathbf{e}_\xi = \frac{1}{K_2}\left(ar\frac{K_3}{K_1}\partial_\varphi Z \partial_\tau + \partial_\xi\right)$$

Connection coefficients

$$\Gamma_{\tau r \tau} = \frac{K_1'}{K_1 K_2}$$

$$\Gamma_{\tau r \xi} =$$

$$\Gamma_{\tau \varphi \xi} = a \frac{K_3}{K_2^2} \left(\qquad \frac{\xi}{2} K_4 \right)$$

$$\Gamma_{\tau \xi r} =$$

$$\Gamma_{r \varphi \tau} = \qquad - \Gamma_{\tau r \varphi}$$

$$\Gamma_{r \varphi \varphi} = \frac{(rK_2)'}{rK_2^2}$$

$$\Gamma_{r \xi r} = \qquad - \frac{\xi}{r} \frac{K_2'}{K_2^2}$$

$$\Gamma_{r \xi \xi} = \frac{K_2'}{K_2^2}$$

$$\Gamma_{\varphi \xi r} =$$

$$\Gamma_{\tau r \varphi} = a \frac{K_3}{K_2^2} \left(1 - \qquad \frac{r}{2} K_4 \right)$$

$$\Gamma_{\tau \varphi r} = -\Gamma_{\tau r \varphi}$$

$$\Gamma_{\tau \xi \tau} = \frac{\xi}{r} \frac{K_1'}{K_1 K_2}$$

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$$\Gamma_{\varphi \xi \tau} = \qquad - \Gamma_{\tau \varphi \xi}$$

$$\Gamma_{\varphi \xi \varphi} = -\frac{\xi}{r} \frac{K_2'}{K_2^2}$$

where $K_4 \equiv (K_3/K_1)(K_1/K_3)'$

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$$\Gamma_{\tau r \tau} = \frac{K_1'}{K_1 K_2}$$
$$\Gamma_{\tau r \xi} = -a\frac{K_3}{K_2^2}\partial_\varphi Z\left(1 - \frac{1}{2r}\left(r^2+\xi^2\right)K_4\right)$$
$$\Gamma_{\tau \varphi \xi} = a\frac{K_3}{K_2^2}\left(Z + \left(1 - \frac{\xi}{r}Z\right)\frac{\xi}{2}K_4\right)$$
$$\Gamma_{\tau \xi r} = -\Gamma_{\tau r \xi}$$
$$\Gamma_{r \varphi \tau} = \frac{\xi}{r}\frac{1}{K_1}\partial_\varphi U - \Gamma_{\tau r \varphi}$$
$$\Gamma_{r \varphi \varphi} = \frac{(rK_2)'}{rK_2^2} - a\xi\frac{K_3}{K_1 K_2}\partial_\varphi U$$
$$\Gamma_{r \xi r} = \frac{W}{K_2} - \frac{\xi}{r}\frac{K_2'}{K_2^2}$$
$$\Gamma_{r \xi \xi} = \frac{K_2'}{K_2^2}$$
$$\Gamma_{\varphi \xi r} = \frac{1}{K_2}\partial_\varphi W$$

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$$\Gamma_{r \varphi r} = \frac{\xi}{r}\frac{1}{K_2}\partial_\varphi W$$
$$\Gamma_{r \xi \tau} = \frac{U}{K_1} - \Gamma_{\tau r \xi}$$
$$\Gamma_{r \xi \varphi} = -ar\frac{K_3}{K_1 K_2}U$$
$$\Gamma_{\varphi \xi \tau} = \frac{1}{K_1}\partial_\varphi U - \Gamma_{\tau \varphi \xi}$$
$$\Gamma_{\varphi \xi \varphi} = -\frac{\xi}{r}\frac{K_2'}{K_2^2} - ar\frac{K_3}{K_1 K_2}\partial_\varphi U$$

where $K_4 \equiv (K_3/K_1)(K_1/K_3)'$

Equations of motion

We use covariant derivative of a vector and covariant divergence of a tensor
$A^i_{;j} = \mathbf{e}_j(A^i) + \Gamma^i_{kj} A^k, \quad A^{ij}_{;j} = \mathbf{e}_j(A^{ij}) + \Gamma^i_{kj} A^{kj} + \Gamma^j_{kj} A^{ik}$

Relativistic hydrodynamics

The law of mass conservation: $(\rho U^i)_{;i} = 0$

The law of energy-momentum conservation: $T^{ik}_{;k} = 0$

$T^{ik} = (\epsilon + p) U^i U^k - p g^{ik} + T^{ik}_\nu - U^i q^k - U^k q^i,$

$g^{ik} = diag\{1, -1, -1, -1\}, \qquad \epsilon = \rho + \epsilon_{th}$

The viscous part is $T^{ik}_\nu = 2\eta\sigma^{ik}$

The shear tensor is $\sigma^{ik} = \frac{1}{2}(U^i_{;j} P^{jk} + U^k_{;j} P^{ji}) - \frac{1}{3} U^j_{;j} P^{ik}$

The projection tensor is $P^{ik} = g^{ik} - U^i U^k$

q^i is the energy flux carried by radiation, η is the dynamical viscosity

A general outline of the derivation

$T^{ik}_{;k}$

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$$T^{ik} = {T_0}^{ik} + {T_1}^{ik} \quad \Longleftarrow \quad U^i = U^i_0 + v^i, \quad \rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \dots$$

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- ~~$\epsilon_{th}, \quad q^i = q^i_0 + q^i_1, \quad \eta_1, \quad p_1$~~

We fix the gauge as $\quad d\xi/d\tau = 0$

A general system of the twist equations

$$\frac{K_2}{K_1}\dot{v}^\varphi + \frac{1}{r}\frac{U^\varphi}{U^\tau}\partial_\varphi v^\varphi + \left(\frac{\partial_r U^\varphi}{U^\tau} + \frac{K_1'}{K_1}\frac{U^\tau}{U^\varphi}\right) v^r + \frac{1}{\rho U^\tau}(\partial_\xi T_\nu^{\varphi\xi} - rW\partial_\xi T_\nu^{r\varphi}) = 0,$$

$$\frac{K_2}{K_1}U^\tau \dot{v}^r + \frac{U^\varphi}{r}\partial_\varphi v^r - 2\frac{K_1'}{K_1 U^\varphi}v^\varphi + \frac{1}{\rho}\partial_\xi T_\nu^{r\xi} = Wr\frac{\partial_\xi p}{\rho},$$

$$\Sigma U^\tau U^\varphi \left\{\partial_\varphi U - a\frac{K_1 K_3}{K_2^2}Z\right\} + \partial_\varphi W \frac{K_1}{K_2}\left\{\Sigma U^\varphi U^r + \bar{T}_\nu^{r\varphi}\right\} =$$

$$- \frac{1}{2r^2 K_2^4}\int d\xi\ \partial_r\,(\,\xi r K_1 K_2^3 U^\varphi \rho \partial_\varphi v^r + r^2 K_1 K_2^3 T_\nu^{r\xi}\,).$$

Σ is the surface mass density

The bar stands for quantities integrated over ξ

$$T_\nu^{r\xi} = -\frac{\eta}{K_2}(\partial_\xi v^r + U^\varphi \partial_\varphi W), \quad T_\nu^{\varphi\xi} = -\frac{\eta}{K_2}\partial_\xi v^\varphi, \quad T_\nu^{r\varphi} = -\eta r\left(\frac{U^\varphi}{rK_2}\right)'$$

A general system of the twist equations $a \ll 1$ & $h/r \ll 1$

$$\frac{K_2}{K_1}\dot{v}^\varphi + \frac{1}{r}\frac{U^\varphi}{U^\tau}\partial_\varphi v^\varphi + \left(\frac{\partial_r U^\varphi}{U^\tau} + \frac{K_1'}{K_1}\frac{U^\tau}{U^\varphi}\right) v^r + \frac{1}{\rho U^\tau}(\partial_\xi T_\nu^{\varphi\xi} - rW\partial_\xi T_\nu^{r\varphi}) = 0,$$

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A reduction of equations

- Consider an isothermal density vertical distribution

$$\rho = \rho_c \exp(-\frac{\xi^2}{2h^2})$$

- The velocity perturbations have now the form

$$v^{\varphi} = \xi(A_1 \sin\varphi + A_2 \cos\varphi) \quad v^r = \xi(B_1 \sin\varphi + B_2 \cos\varphi)$$

- Let us introduce complex notation

$$\mathbf{A} = A_2 + iA_1, \quad \mathbf{B} = B_2 + iB_1 \quad \text{and} \quad \mathbf{W} = \beta e^{i\gamma}$$

- Consider a Novikov-Thorne solution for flat disc as a background and set all time derivatives to zero.

An equation for stationary shapes

$$\frac{K_1}{R^{1/2}D}\frac{d}{dR}\left(\frac{R^{3/2}D}{K_1 U^\tau}f^*(\alpha,R)\frac{d\mathbf{W}}{dR}\right)-3\alpha U^\tau(1-D^{-1})\frac{d\mathbf{W}}{dR}+\frac{4ia}{\delta^2 K_1^3 R^3 U^\varphi}\mathbf{W}=0$$

where $*$ stands for the complex conjugate,

$$f(\alpha,R)=(1+\alpha^2-3i\alpha K_1^2)\,\frac{R(i-\alpha)}{\alpha R(\alpha+2i)-6}+\alpha,$$

$D(R)$ is a Novikov-Thorne correction profile (see Riffert & Herold, 1995),

$\delta=h(R)/R$ is the disc geometrical profile

The solutions have two independent parameters

$$\alpha \quad \rightarrow [0,1]$$

$$\tilde{\delta}=\delta_*/\sqrt{|a|} \quad \rightarrow (0,\infty)$$

Resonant solutions: a self-warping disc or a tilting doll

An analytical solution for the shape of twisted accretion disc for $\alpha = 0$ and $\tilde{\delta} \ll 1$ can be easily constructed using the WKBJ instructions

- The relation between inclination angles at R_{ms} and far from the black hole

$$\mathbf{W}_{\infty} = C_{tot}(\tilde{\delta})\mathbf{W}_0$$

- Average behaviour

$$C_{tot} \sim \tilde{\delta}^{43/30}$$

- $C_{tot} \to 0$ at discrete values of

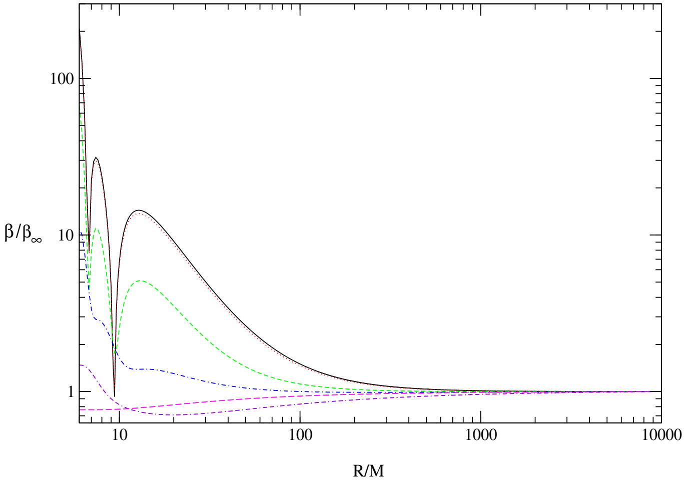
$$\tilde{\delta}_k \simeq \frac{2}{\pi}\,(73/30 + 2k)^{-1}\,,$$

$k = 0, 1, 2, ...$

Numerical results

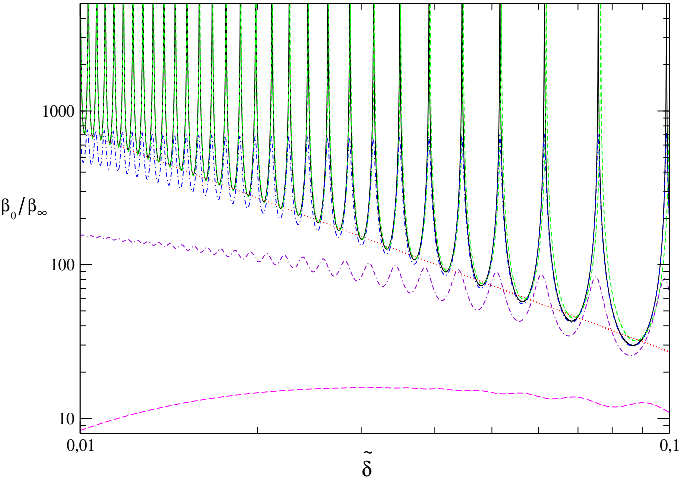
$\tilde{\delta} = 0.1$

$\alpha = 0, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1.0$

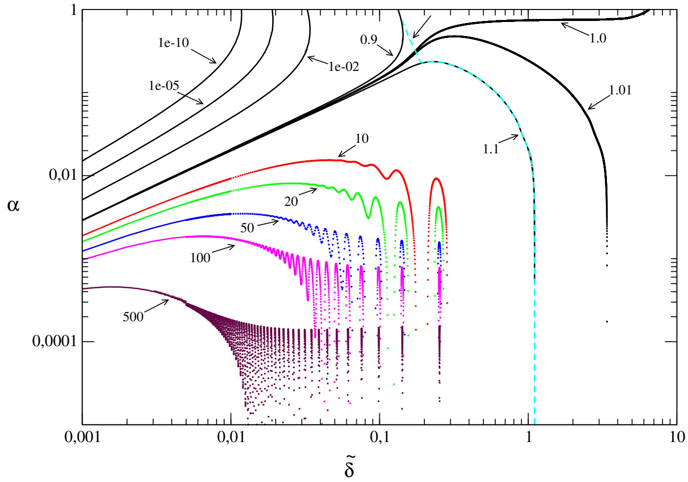


Numerical results

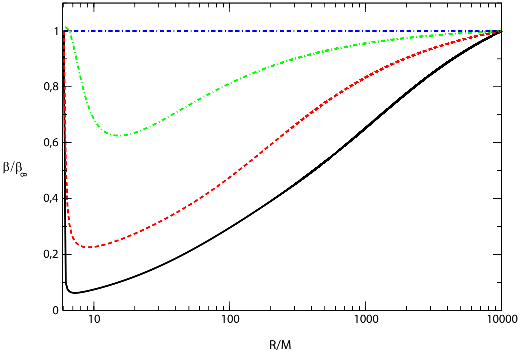
$\alpha = 0, 10^{-4}$, $10^{-3}, 10^{-2}$



Numerical results



Numerical results



$\tilde{\delta} = 10^{-3}, 10^{-2}, 10^{-1}, 1.0$

Physical restrictions

The restriction on the value of the inclination at infinity ($\alpha = 0$) is

- The validity of linear approximation $\beta \ll 1$

$$\mathbf{W}_\infty < \tilde{\delta}^{43/30}$$

- The velocity perturbations should not be too high $v < c_s$

$$\mathbf{W}_\infty < \tilde{\delta}^{49/30}/6$$

A non-zero viscosity $\alpha \neq 0$ usually makes things smoother...

Conclusions

- Dynamical equations describing the evolution and stationary configurations of a fully relativistic thin twisted disc have been derived assuming $\beta \ll 1$ and $a \ll 1$
- For the simple Novikov-Thorne model of a flat disc with a constant value of the Shakura-Sunyaev parameter α equations can be further simplified. The final twist equation for stationary disc is formulated for complex variable **W**. The stationary configurations then can be fully described by two parameters — α & $\tilde{\delta} = \delta_* / \sqrt{|a|}$.
- An analytical theory of stationary disc has been constructed for the case $\alpha = 0$ & $\tilde{\delta} \ll 1$. The disc exhibit prominent oscillations of the inclination angle with R which can grow indefinitely while $\tilde{\delta} \to 0$. Also there are specific “resonant” solutions for discrete values for $\tilde{\delta}$.
- For a moderate value of the viscosity parameter α the Bardeen-Petterson effect is absent. The disc remains to be twisted in the vicinity of a black hole. The disc can align with the equatorial plane of a black hole only in the case of a large value of α and a sufficiently small $\tilde{\delta}$.

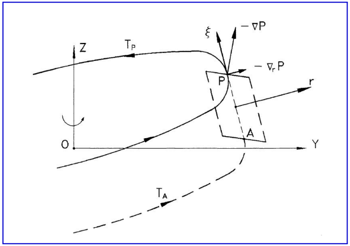
Remarks

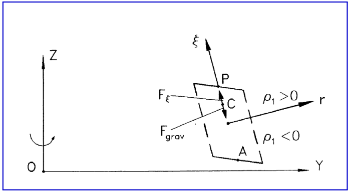
Possible further developments.

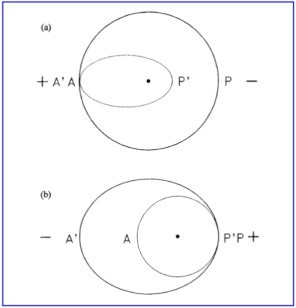
- Time-dependant solutions of twist equations in the relativistic regime. A possibility of quasi-normal modes with $\omega << \Omega$.
- One may include the next order terms in h/r in equations. This would allow to describe twisted slim discs. The problem of singularity near the last stable orbit R_{ms} can be treated in this way.
- Generalisation of stationary solutions to the case $a \sim 1$.
- Feedback effects of warp and twist: self-irradiation of the disc, vertical structure modified by shear velocities etc.
- Calculation of light curves, spectral features and other observational manifestations for particular astrophysical sources.

Overview of previous studies

Comments on the oscillatory regime





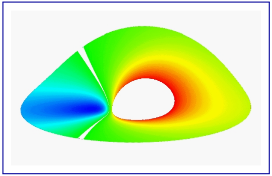


$$\int d\xi \rho_1 \frac{\partial \Phi}{\partial \xi} + \Sigma F_\xi = 0$$

Here, $\frac{\partial \Phi}{\partial \xi} = -\xi \Omega^2$, $\Sigma = \int d\xi \rho_0$.

The background model: geodesic motion & vertical equilibrium

$\beta = 0 \quad \& \quad a = 0$



It is assumed hereafter that the coordinate R is measured in units of M

r-projection of the covariant divergence of the energy-momentum tensor gives

$$U^{\varphi} = (R-3)^{-1/2}, \quad U^{\tau} = \sqrt{1 + (U^{\varphi})^2}$$

$$\Omega = R^{-3/2}$$

ξ-projection of the covariant divergence of the energy-momentum tensor gives

$$\frac{\partial_{\xi} p}{\rho} = -\left(\frac{U^{\varphi}}{r}\right)^2 \xi$$

The background model: viscosity contribution

the law of rest mass conservation gives

$$-\frac{\dot{M}}{2\pi} = \Sigma K_1 K_2^2 r U^r$$

τ & φ-projection of the covariant divergence of the energy-momentum tensor give ("no torque" boundary condition at $R_{ms} = 6$)

$$\bar{\eta} = \frac{\dot{M}}{3\pi} \left(\frac{r^{-1/2}}{U^\tau U^\varphi} \frac{D}{K_1^3 K_2^{3/2}} \right)$$

$$D = 1 - \frac{\sqrt{6}}{\sqrt{R}} - \frac{\sqrt{3}}{2\sqrt{R}} \ln \frac{(\sqrt{R} - \sqrt{3})(3 + 2\sqrt{2})}{(\sqrt{R} + \sqrt{3})}$$

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a viscosity prescription

$$\nu = \alpha K_2 U^\varphi h^2 / r \quad \Longrightarrow \quad \bar{\eta} = \Sigma \nu$$

The background model: viscosity contribution

the law of rest mass conservation gives

$$-\frac{\dot{M}}{2\pi} = \Sigma K_1 K_2^2 r U^r$$

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$$D = 1 - \frac{\sqrt{6}}{\sqrt{R}} - \frac{\sqrt{3}}{2\sqrt{R}} \ln \frac{(\sqrt{R}-\sqrt{3})(3+2\sqrt{2})}{(\sqrt{R}+\sqrt{3})}$$

a viscosity prescription

$$\nu = \alpha K_2 U^\varphi h^2 / r \quad \Longrightarrow \quad \bar{\eta} = \Sigma \nu$$

The background expressions used in the twist equations

$$\int \rho \xi^2 d\xi \simeq \Sigma h^2 = \frac{\dot{M}}{3\pi\alpha}\left(\frac{r^{1/2}}{U^\tau (U^\varphi)^2}\frac{D}{K_1^3 K_2^{5/2}}\right)$$

$$\delta(r) = \delta_* K_1^{3/5} K_2^{1/20} (U^\tau)^{-4/5} D^{1/5} r^{1/20}$$

A reduced system of equations

The “horizontal” part of twist equations can be rewritten as

$$\dot{\mathbf{A}} - (i-\alpha)\Omega\mathbf{A} + \frac{\kappa^2}{2\tilde{\Omega}}\mathbf{B} = -\frac{3}{2}i\alpha K_1 (U^\tau)^2 U^\varphi \Omega \mathbf{W}'$$

$$\dot{\mathbf{B}} - (i-\alpha)\Omega\mathbf{B} - 2\tilde{\Omega}\mathbf{A} = -(i+\alpha)U^\varphi \Omega\, \mathbf{W}'$$

Characteristic frequencies of a slightly perturbed free circular motion in the Schwarzschild metrics are

$$\Omega = R^{-3/2}, \quad \kappa^2 = R^{-3}\left(1 - \frac{6}{R}\right), \quad \tilde{\Omega} = \frac{R-3}{R^2(R-2)^{1/2}}$$

The Newtonian inviscid limit $r \to \infty$, $\alpha \to 0$ is

$$\dot{\mathbf{A}} - i\Omega\mathbf{A} + \frac{\Omega}{2}\mathbf{B} = 0$$

$$\dot{\mathbf{B}} - i\Omega\mathbf{B} - 2\Omega\mathbf{A} = -iU^\varphi \Omega\, \mathbf{W}'$$

$\Longrightarrow$ The Keplerian degeneracy leads to a resonance in the system

A reduced system of equations

The "vertical" part of twist equations can be rewritten as

$$\dot{\mathbf{W}} - i\Omega_{LT}\mathbf{W} + \frac{3}{2}\alpha\delta^2 \frac{K_1^2}{K_2} U^\varphi \left(U^\tau - K_1 (rK_2)^{1/2} \frac{U^\varphi}{D} \right) \mathbf{W}' =$$
$$\frac{\delta^2 K_1^3 U^\varphi}{2r^{1/2} K_2^{3/2} D} \frac{\partial}{\partial r} \left\{ r^{3/2} K_2^{1/2} \frac{D}{K_1^2 U^\tau U^\varphi} (\, (i+\alpha)\mathbf{B} + \alpha U^\varphi \mathbf{W}' \,) \right\}$$

which contains an additional characteristic frequency of the problem, namely, the Lense-Thirring frequency

$$\Omega_{LT} = \Omega - \Omega_\perp = 2aR^{-3}$$

where $\Omega_\perp$ is the frequency of a free vertical harmonic oscillations in the equatorial plane of the Kerr black hole.

Analytical solution: an almost inviscid case

Let us set formally $\alpha = 0$, thus obtaining

$$\frac{d}{dR}\left(b\frac{d}{dR}\mathbf{W}\right) + \lambda\mathbf{W} = 0,$$

where $b \propto D$, $\lambda \propto D/\delta^2$

Close to R_{ms} $\Longrightarrow$ $x = R - R_{ms}$ and $D \simeq x^2/72$, $\delta \propto D^{1/5}$

The twisted disc near $R_{ms} = 3R_g$ is described by

$$\frac{1}{x^2}\frac{d}{dx}x^2\frac{d}{dx}\mathbf{W} + \chi x^{-4/5}\mathbf{W} = 0 \qquad (*)$$

The regular solution of $(*)$ gives a boundary condition at R_{ms}

$$\mathbf{W} = Cx^{-1/2}J_{5/6}(5/3\sqrt{\chi}x^{3/5})$$

The analytical solution: an almost inviscid case

The analytical solution for the shape of twisted accretion disc for $a > 0$ and $\tilde{\delta} \ll 1$

Close to R_{ms}

$$\mathbf{W} = C_1 x^{-1/2} J_{5/6}(5/3\sqrt{\chi} x^{3/5})$$

WKBJ-oscillations in the relativistic region

$$\mathbf{W} \approx \frac{C_2}{(\lambda b)^{1/4}} \cos\left(\int_{R_{ms}}^{R} \sqrt{\lambda/b} dR + \phi_{WKBJ}\right)$$

For $R \gg R_{ms}$

$$\mathbf{W} = C_3 x_1^{3/2} (A_1 J_{-3/5}(C_4 R^{-5/4}) + A_2 J_{3/5}(C_4 R^{-5/4}))$$

Numerical results

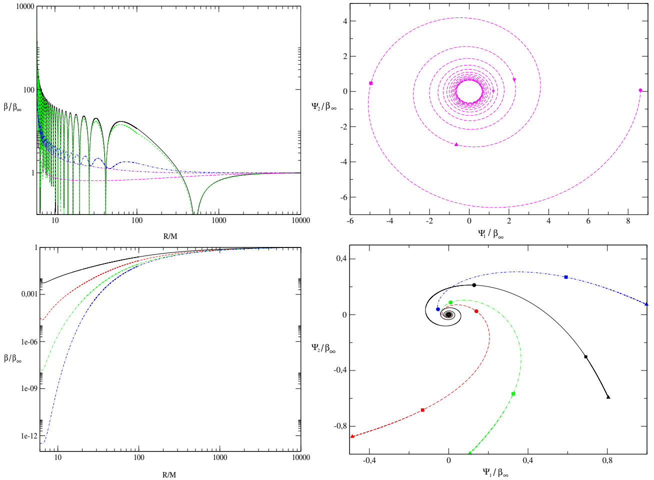
$\Psi_1 = \beta \cos \gamma$
$\Psi_2 = \beta \sin \gamma$

$\tilde{\delta} = 0.01$

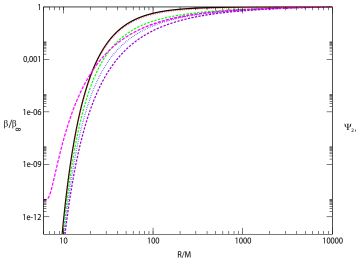
$\alpha = 0, 10^{-4}, 10^{-3}, 10^{-2}$

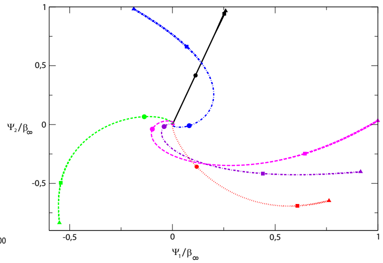
$\tilde{\delta} = 0.01$

$\alpha = 0.05, 0.1, 0.2, 1.0$



Numerical results





$a < 0$

$\tilde{\delta} = 0.01$

$\alpha = 0, 0.01, 0.05, 0.1, 0.2, 1.0$